Quantizations of some Poisson–Lie groups: The bicrossed product construction

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Abstract

By working with several specific Poisson–Lie groups arising from Heisenberg Lie bialgebras and by carrying out their quantizations, a case is made for a useful but simple method of constructing locally compact quantum groups. The strategy is to analyze and collect enough information from a Poisson–Lie group, and using it to carry out a "cocycle bicrossed product construction". Constructions are done using multiplicative unitary operators, obtaining C^*-algebraic, locally compact quantum (semi-)groups.
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1. Introduction

Typically, the most commonly employed method of constructing specific examples of quantum groups is the method of "generators and relations". This is certainly the case in the purely algebraic setting of quantized universal enveloping (QUE) algebras. Even in the

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C^*-algebra setting, compact quantum groups are usually constructed in this way (see, for instance, [26]).

However, when one wishes to construct a non-compact quantum group, this method is not so useful: the generators (essentially the coordinate functions of a group) tend to be unbounded, which gives rise to various technical difficulties. There are ways to handle the difficulties (see [27], where Woronowicz works with the notion of unbounded operators "affiliated" with C^*-algebras), but in general, it is usually better to look for some other methods of construction.

One useful approach not relying on the generators is via the method of deformation quantization. Here, the aim is to deform the (commutative) algebra of functions on a Poisson manifold, in the direction of the Poisson bracket. See [2,25]. [In the C^*-algebra framework, the corresponding notion is the "strict deformation quantization" by Rieffel [19], or its more generalized versions developed later by other authors.] One should recall, however, that this is just a "spatial" deformation, in the sense that the deformation is only for the algebra structure. To obtain a quantum group, one begins with a suitable Poisson–Lie group G (a Lie group equipped with a compatible Poisson bracket) and perform the deformation quantization on the function space $C_0(G)$ – for both its algebra and coalgebra structures.

Some of the non-compact quantum groups obtained by deformation quantization are [20,22,21,29,9]. In these examples, the information at the level of Poisson–Lie groups or Lie bialgebras plays a key role in constructing the quantum groups and their structures. Naturally, there exists a very close relationship between a quantum group obtained in this way and its Poisson–Lie group counterpart.

This last point is quite helpful in working with the quantum group. For instance, as for the example considered by the author in [9], the information from the classical (Poisson) level was useful not only in the construction of the quantum group but also in studying its representation theory, in relation to the dressing orbits [10,13].

On the other hand, despite many advantages, there are some drawbacks to the method of deformation quantization, especially when one wishes to carry it out in the C^*-algebra setting: jumping from the classical level of Poisson–Lie groups to the C^*-algebraic quantum group level is not necessarily an easy task. Even with the guides provided from Poisson data, the actual construction of the structure maps like comultiplication, antipode, or Haar weight often should be done by using different methods. Among the useful tools is the notion of "multiplicative unitary operators" (in the sense of Baaj and Skandalis [1]).

Considering the drawbacks to the geometric approach above, we turn to a more algebraic method of constructing locally compact quantum groups, via the framework of (cocycle) bicrossed products. This goes back to the problem of group extensions in the Kac algebra setting (see [5] for a survey on Kac algebras), and was made systematic by Majid [16,17]. Here, one begins with a certain "matched pair" of groups (or more generally, locally compact quantum groups) and build a larger quantum group as a bicrossed product, possibly with a cocycle. Baaj and Skandalis has a version of this in Section 8 of [1]. For a comprehensive treatment about this framework, see [23].

The best aspect about the bicrossed product method is that it is relatively simple, while sufficiently general to include many special cases. However, as is the case for any general

method, having the framework is not enough to construct actual and specific examples: one needs to have a specific matched pair, together with a compatible cocycle, for this method to work.

So we propose here to combine the advantages of the "geometric" (deformation quantization) method and the "algebraic" framework of cocycle bicrossed products. That is, we first begin with a Poisson–Lie group and analyze its Poisson structure. The Poisson data will help us obtain a suitable matched pair and a compatible cocycle. Then we perform the cocycle bicrossed product construction.

Quantum groups obtained in this way tend to have (twisted) crossed products as their underlying C^*-algebras. And therefore, this program is usually best for constructing solvable-type quantum groups. It is because crossed products often model quantized spaces (for instance, the "Weyl algebra", $C_0(\mathbb{R}^n) \rtimes_\tau \mathbb{R}^n$ with τ being the translation, is the quantized phase space [7]). But with some adjustments, the method could be adopted to construct other types of quantum groups. Meanwhile, having a close connection with the Poisson–Lie group enables us to take advantage of its geometric data in further studying the quantum group as well as in applications.

Our plan in this paper is to illustrate this program through examples, using Poisson–Lie groups associated with several Heisenberg-type Lie bialgebras. Three specific cases are considered. In Case (2), we re-construct our earlier example from [9], which quantizes a certain non-linear Poisson bracket. Case (1) corresponds to a linear Poisson bracket (so a little simpler), and is related with the examples from [20,22,24]. Case (3) is similar to the example given in [6], but is more general. Afterwards, we give more constructions of similar-flavored examples.

To keep the presentation coherent and simple, we will not stray too much away from the Heisenberg-type Lie bialgebras and their quantum counterparts. In this article, the focus is not on giving genuinely new examples, but on showcasing a simple but useful method of constructing specific quantum groups. [Nevertheless, we do obtain a new example below in Case (3) of Section 3.] In this way, we make a case that the geometric, deformation quantization method and the algebraic, bicrossed product method are very much compatible. Examples constructed with the same program but coming from different Poisson–Lie groups will be presented in our future work.

The paper is organized as follows. In Section 2, we discuss some specific Poisson structures coming from various Heisenberg Lie bialgebras. Three specific cases are considered. In Section 3, we carry out the quantizations of the cases from Section 2. By analyzing the Poisson brackets, we obtain, for each of the three cases, a matched pair and a cocycle. These data help us to construct a multiplicative unitary operator, which represents the cocycle bicrossed product construction.

More examples are given in Section 4, by slightly modifying the results obtained in Section 3. Along the way, we will make frequent comparisons between the examples in Sections 3 and 4, and several other examples obtained elsewhere using different methods.

Appendix A shows that the Poisson structures considered in Section 2 actually arise from certain "classical r-matrices". Remembering that a "quantum R-matrix" type operator played a significant role in the representation theory of our earlier example [9,10], this is a useful knowledge.

2. Lie bialgebra structures on a Heisenberg Lie algebra: the Poisson–Lie groups

Let H be the $(2n + 1)$-dimensional Heisenberg Lie group. Its underlying space is $\mathbb{R}^{2n+1}$ and the multiplication on it is given by

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + (x, y')),$$

for $x, x', y, y' \in \mathbb{R}^n$ and $z, z' \in \mathbb{R}$. Here, $\beta(\cdot, \cdot)$ denotes the ordinary inner product.

Its Lie algebra counterpart is the Heisenberg Lie algebra $\mathfrak{h}$. It is generated by the basis elements $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \ldots, n)$, $\mathbf{z}$, with the following relations:

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij}\mathbf{z}, \qquad [\mathbf{z}, \mathbf{x}_i] = [\mathbf{z}, \mathbf{y}_i] = 0.$$

For convenience, we will identify $H \cong \mathfrak{h}$ as vector spaces. This is possible since H is an exponential solvable Lie group (it is actually nilpotent). We will understand that $x = x_1\mathbf{x}_1 + \cdots + x_n\mathbf{x}_n$, and similarly for the other variables. And, we choose a Lebesgue measure on $H \cong \mathfrak{h}$, which is a Haar measure for H.

For a Heisenberg Lie group, all the possible compatible Poisson brackets on it have been classified by Szymczak and Zakrzewski [22]. Among these, we will specifically look at the following, simpler cases (see Definition 2.1 below). The Poisson brackets are described in terms of the cobrackets $\delta : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$, which are one-cocycles with respect to the adjoint representation. It is known from general theory that specifying in such a way a "Lie bialgebra" structure, $(\mathfrak{h}, \delta)$, is equivalent to giving an explicit formula for the Poisson bracket on H (see [15]).

Definition 2.1.

(1) Consider $\delta_1 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ defined by

$$\delta_1(\mathbf{x}_j) = \lambda \mathbf{x}_j \wedge \mathbf{z}, \qquad \delta_1(\mathbf{y}_j) = -\lambda \mathbf{y}_j \wedge \mathbf{z}, \qquad \delta_1(\mathbf{z}) = 0.$$

Here $\lambda \in \mathbb{R}$. To obtain a non-trivial map, we let $\lambda \neq 0$.

(2) Let $\lambda \neq 0$ again, and let $\delta_2 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ be defined by

$$\delta_2(\mathbf{x}_j) = \lambda \mathbf{x}_j \wedge \mathbf{z}, \qquad \delta_2(\mathbf{y}_j) = \lambda \mathbf{y}_j \wedge \mathbf{z}, \qquad \delta_2(\mathbf{z}) = 0.$$

(3) Let (J_{ij}) be a skew, $n \times n$ matrix $(n \geq 2)$, and let $\delta_3 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ be defined by

$$\delta_3(\mathbf{x}_j) = 0, \qquad \delta_3(\mathbf{y}_j) = \sum_{i=1}^{n} J_{ij}\mathbf{x}_i \wedge \mathbf{z}, \qquad \delta_3(\mathbf{z}) = 0.$$

We do not give here an explicit proof that these are indeed Lie bialgebra structures on $\mathfrak{h}$ giving us the compatible Poisson brackets on H. Instead, we can refer to Theorem 2.2 of [22], and in the case of δ_2 above, a careful discussion was given in Section

1 of [9]. See also Appendix A, where we show that they arise from certain "classical *r*-matrices".

Corresponding to each of these Poisson brackets, we can define a Lie bracket on the dual space $\mathfrak{h}^*$ of $\mathfrak{h}$ by $[\cdot, \cdot] = \delta^* : \mathfrak{h}^* \wedge \mathfrak{h}^* \to \mathfrak{h}^*$. That is, $[\mu, \nu]$ is defined by

$$\langle [\mu, \nu], X \rangle = \langle \delta^*(\mu \otimes \nu), X \rangle = \langle \mu \otimes \nu, \delta(X) \rangle, \qquad (*)$$

where $X \in \mathfrak{h}$, $\mu, \nu \in \mathfrak{h}^*$, and $\langle \cdot, \cdot \rangle$ is the dual pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. In this way, we obtain the following "dual" Lie algebra for each of the cases. The proof is straight-forward.

Proposition 2.2. *Let* $\mathfrak{g} = \mathfrak{h}^*$ *be spanned by* $\mathbf{p}_i$, $\mathbf{q}_i$ $(i = 1, \ldots, n)$, $\mathbf{r}$, *which form the dual basis of* $\mathbf{x}_i$, $\mathbf{y}_i$ $(i = 1, \ldots, n)$, $\mathbf{z}$

(1) *On* $\mathfrak{g}$, *define the Lie algebra relations for the basis vectors as follows*:

$$[\mathbf{p}_i, \mathbf{q}_j] = 0, \qquad [\mathbf{p}_i, \mathbf{r}] = \lambda \mathbf{p}_i, \qquad [\mathbf{q}_i, \mathbf{r}] = -\lambda \mathbf{q}_i.$$

    Then $\mathfrak{g}$ *is the Poisson dual of the Lie bialgebra* $(\mathfrak{h}, \delta_1)$.

(2) *On* $\mathfrak{g}$, *define the Lie algebra relations for the basis vectors by*

$$[\mathbf{p}_i, \mathbf{q}_j] = 0, \qquad [\mathbf{p}_i, \mathbf{r}] = \lambda \mathbf{p}_i, \qquad [\mathbf{q}_i, \mathbf{r}] = \lambda \mathbf{q}_i.$$

    This is the Poisson dual of the Lie bialgebra $(\mathfrak{h}, \delta_2)$.

(3) *On* $\mathfrak{g}$, *define the Lie algebra relations by*

$$[\mathbf{p}_i, \mathbf{q}_j] = 0, \qquad [\mathbf{p}_i, \mathbf{r}] = \sum_{j=1}^{n} J_{ij} \mathbf{q}_j, \qquad [\mathbf{q}_i, \mathbf{r}] = 0.$$

    This is the Poisson dual of the Lie bialgebra $(\mathfrak{h}, \delta_3)$.

Each of the dual Lie algebras $\mathfrak{g}$ is actually a Lie bialgebra, whose cobracket $\theta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the dual map of the Lie bracket on $\mathfrak{h}$. This situation is exactly same as in Eq. (*). In other words, $\theta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is defined by its values on the basis vectors of $\mathfrak{g}$ as follows:

$$\theta(\mathbf{p}_i) = 0, \qquad \theta(\mathbf{q}_i) = 0, \qquad \theta(\mathbf{r}) = \sum_{i=1}^{n} (\mathbf{p}_i \otimes \mathbf{q}_i - \mathbf{q}_i \otimes \mathbf{p}_i) = \sum_{i=1}^{n} (\mathbf{p}_i \wedge \mathbf{q}_i).$$

We thus have the (Poisson dual) Lie bialgebra $(\mathfrak{g}, \theta)$, for each of the Heisenberg Lie bialgebras in Definition 2.1. Let us now consider the corresponding Poisson–Lie groups G (dual to the Heisenberg Lie group), together with their Poisson brackets. As before, we will understand $p = p_1 \mathbf{p}_1 + p_2 \mathbf{p}_2 + \cdots + p_n \mathbf{p}_n$, and similarly for the other variables (this explains the notation we use in (3) below).

Proposition 2.3.

(1) *Let G be the $(2n + 1)$-dimensional Lie group, whose underlying space is $\mathbb{R}^{2n+1}$ and the multiplication law is defined by*

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', \ e^{-\lambda r'} q + q', r + r').$$

It is the Lie group corresponding to $\mathfrak{g}$ from Proposition 2.2 *(1). The Poisson bracket on G is given by the expression*

$$\{\phi, \psi\}(p, q, r) = r(\beta(x, y') - \beta(x', y)),$$

for $\phi, \psi \in C^\infty(G)$. Here $\mathrm{d}\phi(p, q, r) = (x, y, z)$ and $\mathrm{d}\psi(p, q, r) = (x', y', z')$, which are naturally considered as elements of $\mathfrak{h}$.

(2) *Let G be the $(2n + 1)$-dimensional Lie group, whose underlying space is $\mathbb{R}^{2n+1}$ and together with the multiplication law*

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', \ e^{\lambda r'} q + q', r + r').$$

It is the Lie group corresponding to $\mathfrak{g}$ from Proposition 2.2 *(2). The Poisson bracket on G is given by*

$$\{\phi, \psi\}(p, q, r) = \left(\frac{e^{2\lambda r} - 1}{2\lambda} \right) (\beta(x, y') - \beta(x', y)),$$

for $\phi, \psi \in C^\infty(G)$. Here again, we use the natural identification of $\mathrm{d}\phi(p, q, r) = (x, y, z)$ and $\mathrm{d}\psi(p, q, r) = (x', y', z')$ as elements of $\mathfrak{h}$.

(3) *Let G be the $(2n + 1)$-dimensional Lie group, together with the multiplication law*

$$(p, q, r)(p', q', r') = \left(p + p', q + q' + r' \sum_{i, j=1}^{n} J_{ij} p_i \mathbf{q}_j, r + r' \right).$$

For it to be non-trivial, we need $n \geq 2$. This gives us the Lie group corresponding to $\mathfrak{g}$ from Proposition 2.2 *(3). The Poisson bracket on G is given by*

$$\{\phi, \psi\}(p, q, r) = r(\beta(x, y') - \beta(x', y)) + \frac{r^2}{2} \sum_{k, j=1}^{n} J_{kj}(y_j y_k' - y_k y_j'),$$

for $\phi, \psi \in C^\infty(G)$. Again, $\mathrm{d}\phi(p, q, r) = (x, y, z)$ and $\mathrm{d}\psi(p, q, r) = (x', y', z')$, viewed as elements of $\mathfrak{h}$.

Proof.   Constructing G from $\mathfrak{g}$ is rather straightforward. In each of the three cases, G is a (connected and simply connected) exponential solvable Lie group corresponding to $\mathfrak{g}$.

As before, we can identify $G \cong \mathfrak{g}$ as vector spaces. Note that the definitions of the group multiplications are chosen in such a way that an ordinary Lebesgue measure becomes the Haar measure for G (in particular, for Case (2)).

To find the expression for the Poisson bracket, we follow the standard procedure: First, consider Ad : $G \to \mathrm{Aut}(\mathfrak{g})$, the adjoint representation of G on $\mathfrak{g}$. We then look for a map $F : G \to \mathfrak{g} \wedge \mathfrak{g}$, that is a group one-cocycle on G for the Ad-representation and whose derivative at the identity element, $\mathrm{d}F_e$, coincides with θ above. Note that since θ depends only on the r-variable, so should F. In other words, we look for a map F such that:

$$F(r + r') = F(r) + \mathrm{Ad}_{(0,0,r)} F(r'), \qquad \mathrm{d}F_{(0,0,0)}(r) = \theta(r) = r \sum_{k=1}^{n} (\mathbf{p}_k \wedge \mathbf{q}_k).$$

Once we have the one-cocycle F, the Poisson bivector field is then obtained by the right translation of F.

It is true that integrating θ to F is not always easy. However, it is not too difficult in our three cases above, due to our Lie bialgebra structures being rather simple. In particular, for Case (2), the computation was given in the proof of Theorem 2.2 in [9]. Case (1) is similar but easier, since the map F (and the Poisson bracket) is linear.

As for Case (3), note first that the representation Ad sends the basis vectors of $\mathfrak{g}$ as follows:

$$\mathrm{Ad}_{(0,0,r)}(\mathbf{p}_k) = (0, 0, r)\mathbf{p}_k(0, 0, -r) = \mathbf{p}_k - r \sum_{j=1}^{n} J_{kj}\mathbf{q}_j,$$

$$\mathrm{Ad}_{(0,0,r)}(\mathbf{q}_k) = \mathbf{q}_k, \qquad \mathrm{Ad}_{(0,0,r)}(\mathbf{r}) = \mathbf{r}.$$

Considering the requirements for the map F given above, we obtain the following expression for F:

$$F(p, q, r) = F(r) = r \sum_{k=1}^{n} (\mathbf{p}_k \wedge \mathbf{q}_k) - \frac{r^2}{2} \sum_{k,j=1}^{n} (J_{kj}\mathbf{q}_j \wedge \mathbf{q}_k).$$

Since the right translations are $R_{(p,q,r)_*}(\mathbf{p}_k) = \mathbf{p}_k + r \sum_{j=1}^{n} J_{kj}\mathbf{q}_j$ and since $R_{(p,q,r)_*}(\mathbf{q}_k) = \mathbf{q}_k$, we thus have the expression for our Poisson bracket:

$$\{\phi, \psi\}(p, q, r) = \langle R_{(p,q,r)_*} F(p, q, r), \, \mathrm{d}\phi(p, q, r) \wedge \mathrm{d}\psi(p, q, r)\rangle$$

$$= r(\beta(x, y') - \beta(x', y)) + \frac{r^2}{2} \sum_{k,j=1}^{n} J_{kj}(y_j y'_k - y_k y'_j). \quad \square$$

Remark 2.4. Cases (1) and (2) look almost the same, and the difference may look rather innocent. However, Case (1) gives us a linear Poisson bracket, while Case (2) is non-linear. Note also that the group G is unimodular in Case (1), while G is non-unimodular in Case (2). It turns out that Case (2) is technically deeper, while having richer properties: for instance, in working with the Haar weight and in representation theory of its quantum group counterpart

(see [9,10,12,13]). Meanwhile, Case (3) gives another non-linear Poisson bracket (in this situation, G is unimodular).

3. Construction of quantum groups

Now that we have described our Poisson–Lie groups, let us construct their quantum group counterparts. But first, we should mention that these cases are not totally new, having been studied elsewhere previously (though Case (3) will be new). As we noted in Section 1, our real focus is on illustrating our improved approach of using the Poisson data to obtain quantum groups, via "cocycle bicrossed products".

Considering this, it will be sufficient to just describe appropriate multiplicative unitary operators. As is known in general theory [1,28,14], having a "regular" (or more generally, "manageable") multiplicative unitary operator gives rise to a C^*-bialgebra, which is really a quantum semigroup. As for the theory on cocycle bicrossed products in the C^*-algebraic quantum group setting, we refer [23]. On the other hand, since we are planning to work with multiplicative unitary operators, our approach will be actually closer to that given in Section 8 of [1].

In the below, we treat separately the three cases we described in the previous section. Using the Poisson geometric data as a guide, we will find a suitable matched pair and a compatible cocycle. We will use this information to construct a multiplicative unitary operator, giving rise to a C^*-bialgebra having the structure of a cocycle bicrossed product.

3.1. Cases (1) and (2)

Finding the multiplicative unitary operators for Cases (1) and (2) goes essentially the same way. Since Case (2) is the more complicated one between the two, we will look at this case more carefully. Compare with the construction procedure given in Sections 2 and 3 of [9], where the approach relied much more on Poisson geometry and the deformation process.

Since we will use the non-linear expression $(e^{2\lambda r} - 1)/2\lambda$ quite often, let us give it the special notation, $\eta_\lambda(r)$. Note that if $\lambda = 0$, it degenerates into the linear expression $\eta_{\lambda=0}(r) = r$.

From now on, let the group G and the Poisson bracket on it be as described in Proposition 2.3 (2). It is the dual Poisson–Lie group of H, corresponding to the Lie bialgebra $(\mathfrak{h}, \delta_2)$. Meanwhile, let $Z = \{(0, 0, z) : z \in \mathbb{R}\}$ be the center of H. Its Lie algebra counterpart is denoted by $\mathfrak{z}(\subseteq \mathfrak{h})$. From the expression of the Poisson bracket, we obtain the following continuous field of group cocycles for H/Z.

Proposition 3.1. *Let $r \in \mathfrak{h}^*/\mathfrak{z}^\perp$. Define the map $\sigma^r : H/Z \times H/Z \to \mathbb{T}$ by*

$$\sigma^r((x, y), (x', y')) = \bar{e}[\eta_\lambda(r)\beta(x, y')],$$

where $e(t) = e^{2\pi i t}$, *so* $\bar{e}(t) = e^{-2\pi i t}$. *Then each* σ^r *is a* $\mathbb{T}$-*valued, normalized group cocycle for* H/Z. *Moreover,* $r \mapsto \sigma^r$ *forms a continuous field of cocycles.*

Remark 3.2. Verifying the cocycle identity is straightforward, and the continuity is also clear. The point is that our Poisson bracket can be written as a sum of the (trivial) linear Poisson bracket on $(\mathfrak{h}/\mathfrak{z})^*$ and the map ω, where $\omega : ((x, y), (x', y')) \mapsto \eta_\lambda(r)((x, y'), \beta(x', y))$ is a Lie algebra cocycle on $\mathfrak{h}/\mathfrak{z}$ having values in $C^\infty(\mathfrak{h}^*/\mathfrak{z}^\perp)$. We then obtain the group cocycle σ above, by "integrating" ω. In a more general setting, this procedure of finding a group cocycle from a Poisson bracket is discussed in [8] (see, in particular, the discussion from Theorem 2.2 to Proposition 3.3 in that paper).

In addition to giving us the group cocycle σ, the Poisson bracket strongly suggests us to work with the $(x, y; r)$ variables, where $(x, y) \in H/Z$ and $r \in \mathfrak{h}^*/\mathfrak{z}^\perp$. Dual space to H/Z is $(\mathfrak{h}/\mathfrak{z})^* = \mathfrak{z}^\perp$, whose typical element is denoted by (p, q). Let us take this suggestion and break the group G into two, obtaining the following matched pair.

Definition 3.3. Let G_1 and G_2 be subgroups of G, defined by

$$G_1 = \{(0, 0, r) : r \in \mathbb{R}\}, \qquad G_2 = \{(p, q, 0) : p, q \in \mathbb{R}^n\}.$$

Clearly, as a space $G \cong G_2 \times G_1$. Moreover, G_1 and G_2 are closed subgroups of G, such that $G_1 \cap G_2 = \{(0, 0, 0)\}$. And, any element (p, q, r) of G can be (uniquely) expressed as a product: $(p, q, r) = (0, 0, r)(p, q, 0)$, with $(0, 0, r) \in G_1$ and $(p, q, 0) \in G_2$. In other words, the groups G_1 and G_2 form a *matched pair*.

From the matched pair, we naturally obtain the group actions $\alpha : G_1 \times G_2 \to G_2$ and $\gamma : G_2 \times G_1 \to G_1$, defined by

$$\alpha_r(p, q) := (e^{-\lambda r} p, \ e^{-\lambda r} q), \qquad \gamma_{(p,q)}(r) := r.$$

Here we are using the obvious identification of (p, q) with $(p, q, 0)$, and similarly for r and $(0, 0, r)$. Note that these actions are defined so that we have: $(\alpha_r(p, q))(\gamma_{(p,q)}(r)) = (e^{-\lambda r} p, \ e^{-\lambda r} q, 0)(0, 0, r) = (p, q, r)$.

Let us now convert the information we obtained so far into the language of Hilbert space operators and operator algebras. To begin with, let us fix a Lebesgue measure on $H(= \mathfrak{h})$, which is the Haar measure for H. And on $G(= \mathfrak{g})$, which is considered as the dual vector space of H, we give the dual Lebesgue measure. (As noted earlier, this will be again the Haar measure for G.) They are chosen so that the Fourier transform becomes the unitary operator (from $L^2(H)$ to $L^2(G)$), and the Fourier inversion theorem holds. Similarly, "partial" Fourier transform can be considered: for instance, between functions in the $(p, q; r)$ variables and those in the $(x, y; r)$ variables. See Remark 1.7 of [9].

Following Baaj and Skandalis [1], the information about the groups G_1 and G_2 can be incorporated into certain multiplicative unitary operators X and Y. The result is given below. Note that we are also expressing our operators in the $(x, y; r)$ variables, so that we can later work within that setting.

Proposition 3.4. *Let $X \in \mathcal{B}(L^2(G_1 \times G_1))$ and $Y \in \mathcal{B}(L^2(G_2 \times G_2))$ be defined such that for $\xi \in L^2(G_1 \times G_1)$ and $\zeta \in L^2(G_2 \times G_2)$, we have*:

$$X\xi(r; r') = \xi(r + r'; r'), \qquad Y\zeta(p, q; p', q') = \zeta(p + p', q + q'; p', q').$$

They are multiplicative unitary operators. Meanwhile, by Fourier transform, Y can be expressed as an operator contained in $\mathcal{B}(L^2(H/Z \times H/Z))$, in the (x, y) variables. This means that we are regarding $\mathcal{F}^{-1}Y\mathcal{F}$ as same as Y, for convenience. It then reads:

$$Y\zeta(x, y; x', y') = \zeta(x, y; x' - x, y' - y), \qquad \zeta \in L^2(H/Z).$$

We have:

$$C_0(G_1) \cong \overline{\{(\omega \otimes \mathrm{id})(X) : \omega \in \mathcal{B}(L^2(G_1))_*\}}^{\|\ \|} \subseteq \mathcal{B}(L^2(G_1)),$$

$$C^*(G_1) \cong \overline{\{(\mathrm{id} \otimes \omega)(X) : \omega \in \mathcal{B}(L^2(G_1))_*\}}^{\|\ \|} \subseteq \mathcal{B}(L^2(G_1)),$$

$$C_0(G_2) \cong C^*(H/Z) \cong \overline{\{(\omega \otimes \mathrm{id})(Y) : \omega \in \mathcal{B}(L^2(H/Z))_*\}}^{\|\ \|} \subseteq \mathcal{B}(L^2(H/Z)),$$

$$C^*(G_2) \cong C_0(H/Z) \cong \overline{\{(\mathrm{id} \otimes \omega)(Y) : \omega \in \mathcal{B}(L^2(H/Z))_*\}}^{\|\ \|} \subseteq \mathcal{B}(L^2(H/Z)).$$

Proof. We are just following [1]. For the expression of the operator $Y \in \mathcal{B}(L^2(H/Z \times H/Z))$, we used the Fourier inversion theorem. Since the groups are abelian, all the computations are quite simple. $\square$

By the result of Proposition 3.4, a function $f \in C_0(G_1)$ is considered same as the multiplication operator $L_f \in \mathcal{B}(L^2(G_1))$, defined by $L_f\xi(r) = f(r)\xi(r)$. Similar for $g \in C_0(G_2)$, which is also considered as the multiplication operator $\lambda_g \in \mathcal{B}(L^2(G_2))$. In the (x, y) variables, this is equivalent to saying that for $g \in C_c(H/Z) \subseteq C^*(H/Z)$, the operator $L_g \in \mathcal{B}(L^2(H/Z))$ is such that for $\zeta \in L^2(H/Z)$, we have:

$$L_g\zeta(x, y) = \int g(\tilde{x}, \tilde{y})\zeta(x - \tilde{x}, y - \tilde{y})\, d\tilde{x}\, d\tilde{y}.$$

At the level of the C^*-algebras $C_0(G_1)$ and $C_0(G_2)$, the group actions α and γ we obtained earlier (though γ is trivial) are expressed as coactions $\alpha : C_0(G_2) \to M(C_0(G_2) \otimes C_0(G_1))$ and $\gamma : C_0(G_1) \to M(C_0(G_2) \otimes C_0(G_1))$, given by

$$\alpha(g)(p, q; r) = g(\mathrm{e}^{-\lambda r}p,\ \mathrm{e}^{-\lambda r}q) = g(\alpha_r(p, q)),$$

$$\gamma(f)(p, q; r) = f(r) = f(\gamma_{(p,q)}(r)).$$

Furthermore, the coactions α and γ can be realized using a certain unitary operator Z, as follows.

Proposition 3.5. *Let $Z \in \mathcal{B}(L^2(G)) = \mathcal{B}(L^2(G_2 \times G_1))$ be defined by*

$$Z\xi(p, q; r) = (\mathrm{e}^{-\lambda r})^n\xi(\mathrm{e}^{-\lambda r}p,\ \mathrm{e}^{-\lambda r}q; r).$$

Then we have, for $g \in C_0(G_2)$ and $f \in C_0(G_1)$,

$$Z(\lambda_g \otimes 1)Z^* = (\lambda \otimes L)(\alpha(g)), \qquad Z(1 \otimes L_f)Z^* = (\lambda \otimes L)(\gamma(f)).$$

Proof. A straightforward computation shows that for $\xi \in L^2(G)$

$$Z(\lambda_g \otimes 1)Z^*\xi(p, q, r) = g(\mathrm{e}^{-\lambda r}p, \ \mathrm{e}^{-\lambda r}q)\xi(p, q, r) = (\lambda \otimes L)(\alpha(g))\xi(p, q, r).$$

And similarly, $Z(1 \otimes L_f)Z^* = (\lambda \otimes L)(\gamma(f))$. $\square$

Remark 3.6. By using the operator realizations $g = \lambda_g$ and $f = L_f$, as well as $\alpha(g) = (\lambda \otimes L)(\alpha(g))$ and $\gamma(f) = (\lambda \otimes L)(\gamma(f))$, we may simply write the above result as: $\alpha(g) = Z(g \otimes 1)Z^*$ and $\gamma(f) = Z(1 \otimes f)Z^*$.

Since we prefer to work with the $(x, y; r)$ variables, it will be more convenient to introduce the Hilbert space $\mathcal{H} := L^2(H/Z \times G_1)$, consisting of the L^2-functions in the $(x, y; r)$ variables. Then by considering that $G_2 = (H/Z)^*$, or equivalently that $C_0(G_2) \cong C^*(H/Z)$, we may as well regard the coactions α and γ to be on $C^*(H/Z)$ and $C_0(G_1)$. (In that case, the definitions of α and γ should be modified accordingly.) In this setting, the operator Z will become $Z \in \mathcal{B}(\mathcal{H})$, defined by

$$Z\xi(x, y; r) = (\mathrm{e}^{\lambda r})^n \xi(\mathrm{e}^{\lambda r}x, \ \mathrm{e}^{\lambda r}y; r).$$

The multiplicative unitary operator associated to the matched pair (G_1, G_2) is given in the next proposition. It gives us two C^*-bialgebras, which are the bicrossed product algebras coming from the matched pair. Note that at this moment, the cocycle is not considered yet.

Proposition 3.7. *Let $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(H/Z \times G_1 \times H/Z \times G_1))$ be the unitary operator defined by*

$$V = (Z_{12}X_{24}Z_{12}^*)Y_{13},$$

where we are using the standard leg notation. More specifically

$$V\xi(x, y, r; x', y', r') = (\mathrm{e}^{-\lambda r'})^n \xi(\mathrm{e}^{-\lambda r'}x, \ \mathrm{e}^{-\lambda r'}y, r + r'; x' - \mathrm{e}^{-\lambda r'}x, \ y' - \mathrm{e}^{-\lambda r'}y, r').$$

It is multiplicative, and by considering the "left [and right] slices" of V, we obtain the following two C^*-algebras, as contained in the operator algebra $\mathcal{B}(\mathcal{H})$:

$$A_V = \{(\omega \otimes \mathrm{id}_{\mathcal{H}})(V) : \omega \in \mathcal{B}(\mathcal{H})_*\} \cong C_0(G_1) \rtimes_\gamma (H/Z),$$

$$\hat{A}_V = \{(\mathrm{id}_{\mathcal{H}} \otimes \omega)(V) : \omega \in \mathcal{B}(\mathcal{H})_*\} \cong C_0(H/Z) \rtimes_\alpha G_1.$$

Here the actions are defined by $\gamma_{(x,y)}(r) := r$ (trivial action), and $\alpha_r(x, y) := (\mathrm{e}^{\lambda r}x, \ \mathrm{e}^{\lambda r}y)$. The algebras A_V and $\hat{A}_V$ are actually C^*-bialgebras, whose comultiplications are given by $\Delta_V(a) = V(a \otimes 1)V^*$ for $a \in A_V$, and $\hat{\Delta}_V(b) = V^*(1 \otimes b)V$ for $b \in \hat{A}_V$.

Remark 3.8. The choice of the operator V is suggested from Section 8 of [1], where discussions are given on obtaining multiplicative unitary operators from a matched pair (*couple assorti*). The point is that the operators X and Y encode the groups G_1 and G_2, while the operator Z carries the information about the actions α and γ. Indeed, the statement above concerning the characterizations for A_V and $\hat{A}_V$ is a fairly general result.

Proof. We skip the proof that V is multiplicative, though a direct verification of the multiplicativity is not really difficult. Instead, we point out that V is actually a degenerate case of the multiplicative unitary operator U given in [9] (see proof of Proposition 3.9 below). Once it is known that V is indeed multiplicative, the general theory assures us that we have a pair of C^*-bialgebras A_V and $\hat{A}_V$, contained in $\mathcal{B}(\mathcal{H})$.

Meanwhile, it is also not difficult to show directly that as a C^*-algebra, we have: $A_V \cong \overline{L(C_c(H/Z \times G_1))}^{\| \|}$, where

$$L_f \xi(x, y; r) = \int f(\tilde{x}, \tilde{y}; r)\xi(x - \tilde{x}, y - \tilde{y}; r)\, d\tilde{x}\, d\tilde{y},$$

for $f \in C_c(H/Z \times G_1)$ and $\xi \in \mathcal{H}$. It then follows that: $A_V \cong C_0(G_1) \rtimes (H/Z)$, which is the crossed product algebra with the trivial action.

Similarly, we can also show that $\hat{A}_V \cong \overline{\rho(C_c(H/Z \times G_1))}^{\| \|}$, where

$$\rho_f \xi(x, y; r) = \int f(x, y; \tilde{r})\xi(e^{\lambda \tilde{r}}x,\ e^{\lambda \tilde{r}}y; r - \tilde{r})\, d\tilde{r}$$

$$= \int f(x, y; \tilde{r})\xi(\alpha_{\tilde{r}}(x, y); r - \tilde{r})\, d\tilde{r},$$

for $f \in C_c(H/Z \times G_1)$ and $\xi \in \mathcal{H}$. From this, we see that as a C^*-algebra, $\hat{A}_V \cong C_0(H/Z) \rtimes_\alpha G_1$.

We did not provide explicit computations here, but see the comments made in proof of Proposition 3.9. The C^*-algebras A_V and $\hat{A}_V$ considered here are actually degenerate cases of the ones in that proposition. See also the proof of Proposition 3.12, which has a similar result. $\square$

Now that we have the multiplicative unitary operator V encoding the matched pair (G_1, G_2), our final task is to incorporate the cocycle term. In the setting of multiplicative unitary operators, we need to look for a function $\Theta : (H/Z \times G_1) \times (H/Z \times G_1) \to \mathbb{T}$ such that $V\Theta$ is still multiplicative (see Section 8 of [1]). Note here that we are regarding Θ as a unitary operator such that $\Theta\xi(x, y, r; x', y', r') = \Theta(x, y, r; x', y', r')\xi(x, y, r; x', y', r')$. Motivated by Proposition 3.1, let us take Θ to be the map

$$\Theta(x, y, r; x', y', r') := \bar{e}[\eta_\lambda(r')\beta(x, y')].$$

As the next proposition shows, we obtain in this way a multiplicative unitary operator $V_\Theta = V\Theta$. It determines a pair of C^*-bialgebras that are realized as cocycle bicrossed products.

Proposition 3.9 ([Quantization of Case (2)]). *Let Θ and V be as above. Then the operator $V_\Theta := V\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ is a multiplicative unitary operator. Specifically*

$$V_\Theta \xi(x, y, r; x', y', r') = (e^{-\lambda r'})^n \bar{e}[\eta_\lambda(r')\beta(e^{-\lambda r'}x, y' - e^{-\lambda r'}y)]$$

$$\xi(e^{-\lambda r'}x, \ e^{-\lambda r'}y, r + r'; x' - e^{-\lambda r'}x, y' - e^{-\lambda r'}y, r').$$

The C^*-bialgebras associated with V_Θ are:

$$A \cong C_0(G_1) \rtimes_\gamma^\sigma (H/Z), \qquad \hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1,$$

where the comultiplications are given by $\Delta(a) = V_\Theta(a \otimes 1)V_\Theta^*$ for $a \in A$, and $\hat{\Delta}(b) = V_\Theta^*(1 \otimes b)V_\Theta$ for $b \in \hat{A}$.

Proof. The operator V_Θ coincides with the multiplicative unitary operator U obtained in Proposition 3.1 of [9]. We will refer to that paper for the proof of the multiplicativity. If $\Theta \equiv 1$, the operator V_Θ degenerates into V given in Proposition 3.7, giving us the proof of its multiplicativity we skipped.

As for the characterization of the C^*-algebra A as a twisted crossed product algebra (in the sense of [18]), see Proposition 2.2 of [12], as well as [9]. As noted earlier, γ is actually a trivial cocycle, while σ is the group cocycle for H/Z defined in Proposition 3.1. In case $\sigma \equiv 1$ (corresponding to $\Theta \equiv 1$), it will degenerate to $A_V \cong C_0(G_1) \rtimes (H/Z)$ in Proposition 3.7.

The characterization for the C^*-algebra $\hat{A}$ can be found in Proposition 2.2 of [11]. Note that $\hat{A}$ does not change from the case without the cocycle, given in Proposition 3.7 (so $\hat{A} \cong \hat{A}_V$). Only its comultiplication changes, by $\hat{\Delta}(b) = \Theta^* \hat{\Delta}_V(b)\Theta$. $\qquad\square$

Our approach was different, but since we obtained the same multiplicative unitary operator as in [9], we can use the result of that paper (as well as [12]) to construct the rest of the quantum group structure for (A, Δ). It is a "quantized $C_0(G)$", as well as a "quantized $C^*(H)$". (Note that $A \cong C^*(H)$, if $\lambda = 0$.) It is the quantization of the Poisson–Lie group G given in Proposition 2.3 (2).

Reflecting the fact that the group G was non-unimodular, the quantum group (A, Δ) turns out to be also non-unimodular (see [12]). See also [10,13], where we take advantage of the close relationship between (G, H) and (A, Δ) to discuss the representation theory of A. For instance, a "quasitriangular, quantum R-matrix" type operator can be found, corresponding to the classical r-matrix given in Appendix A.

Meanwhile, $(\hat{A}, \hat{\Delta})$ is the dual quantum group of (A, Δ). This is studied in [11], and may be regarded as a "quantized $C_0(H)$", or a "quantized $C^*(G)$". As H is unimodular, so is $(\hat{A}, \hat{\Delta})$. This is a "quantum Heisenberg group", but is different from the one constructed in [22] or in [24]. See below.

Since we are satisfied with Case (2), let us now turn our attention to Case (1). Consider the Poisson–Lie group G and the Poisson bracket on it as described in Proposition 2.3 (1).

Noting the similarity with Case (2), and with a slight modification of the procedure, we obtain the following.

Proposition 3.10 ([Quantization of Case (1)]).

(1) *Let G_1 and G_2 be defined by*

$$G_1 = \{r : r \in \mathbf{R}\}, \qquad G_2 = \{(p, q) : p, q \in \mathbf{R}^n\}.$$

Consider also the group actions $\alpha : G_1 \times G_2 \to G_2$ and $\gamma : G_2 \times G_1 \to G_1$, given by

$$\alpha_r(p, q) := (\mathrm{e}^{-\lambda r} p, \, \mathrm{e}^{\lambda r} q), \qquad \gamma_{(p,q)}(r) := r.$$

In this way, we obtain the matched pair (G_1, G_2).

(2) *Let $X \in \mathcal{B}(L^2(G_1 \times G_1))$ and $Y \in \mathcal{B}(L^2(H/Z \times H/Z))$ be the operators defined by $X\xi(r; r') = \xi(r + r'; r')$, and by $Y\xi(x, y; x', y') = \xi(x, y; x' - x, y' - y)$. In addition, let $Z \in \mathcal{B}(L^2(H/Z \times G_1))$ be such that $Z\xi(x, y; r) = \xi(\mathrm{e}^{\lambda r}x, \, \mathrm{e}^{-\lambda r}y; r)$, and let $\Theta(x, y, r; x', y', r') := \bar{e}[r'\beta(x, y')]$, which is considered as a unitary operator. Then $V_\Theta := (Z_{12}X_{24}Z_{12}^*Y_{13})\Theta$ is a multiplicative unitary operator contained in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(H/Z \times G_1 \times H/Z \times G_1))$.*

(3) *The C^*-bialgebras associated with V_Θ are:*

$$A \cong C_0(G_1) \rtimes_\gamma^\sigma (H/Z) \cong C^*(H), \qquad \hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1,$$

together with the comultiplications $\Delta(a) := V_\Theta(a \otimes 1)V_\Theta^*$ for $a \in A$, and $\hat{\Delta}(b) := V_\Theta^*(1 \otimes b)V_\Theta$ for $b \in \hat{A}$.

Remark 3.11. The proof is done in exactly the same way as in the earlier part of this section, concerning Case (2). Similarly to Case (2), the C^*-algebra A is isomorphic to a twisted crossed product algebra, with the twisting cocycle $\sigma^r((x, y), (x', y')) := \bar{e}[r\beta(x, y')]$. But by using (partial) Fourier transform, it can be shown easily that $A \cong C^*(H)$. This does not hold in Case (2). It reflects the fact that in Case (1), unlike in Case (2), the Poisson bracket we began with is a linear Poisson bracket (dual to the Lie bracket on H).

Again the approach was different, but we point out here that $(\hat{A}, \hat{\Delta})$ from Proposition 3.10 is isomorphic to the example given by Van Daele in [24], as well as to the one given by Szymczak and Zakrzewski [22]. Meanwhile, (A, Δ) is its dual counterpart. This is actually the special case of the example considered by Rieffel in [20] (see also Section 4.2 below). These three examples are considered to be among the pioneering works on non-compact quantum groups. We will refer to these other papers for the construction of the rest of quantum group structures for $(\hat{A}, \hat{\Delta})$ and for (A, Δ).

Representation theory for Case (1) has not been done in the literatures, but actually, due to the fact that it corresponds to a linear Poisson bracket and also to a triangular classical r-matrix (see Appendix A), it is much simpler than that of Case (2). Meanwhile, reflecting

the fact that both H and G are unimodular, both $(\hat{A}, \hat{\Delta})$ and (A, Δ) for Case (1) turns out to be unimodular (i.e. their Haar weights are both right and left invariant).

3.2. Case (3)

Let us now consider the case of the Lie group G and the Poisson bracket on it as described in Proposition 2.3 (3). Analogously to Definition 3.3 and Proposition 3.10 (1), we will begin with the matched pair (G_1, G_2). Here, the groups are

$$G_1 = \{r : r \in \mathbb{R}\}, \qquad G_2 = \{(p, q) : p, q \in \mathbb{R}^n\},$$

together with the group actions $\alpha : G_1 \times G_2 \to G_2$ and $\gamma : G_2 \times G_1 \to G_1$, given by $\alpha_r(p, q) := \left(p, q - r \sum_{i,j} J_{ij} p_i \mathbf{q_j} \right)$ and $\gamma_{(p,q)}(r) := r$.

Note that, as before, $G \cong G_1 \times G_2$ as a space, while G_1 and G_2 may be regarded as closed subgroups of G such that $G_1 \cap G_2 = \{(0, 0, 0)\}$. This is done by viewing $(0, 0, r)$ and $(p, q, 0)$ as same as $r \in G_1$ and $(p, q) \in G_2$, respectively. Any element of G can be (uniquely) expressed as a product: $(p, q, r) = (0, 0, r)(p, q, 0)$. The actions are defined so that we have: $(\alpha_r(p, q))(\gamma_{(p,q)}(r)) = (p, q, r)$.

We will again work with the $(x, y; r)$ variables, in $H/Z \times G_1$. So the multiplicative unitary operators associated with the groups G_1 and G_2 are $X \in \mathcal{B}(L^2(G_1 \times G_1))$ and $Y \in \mathcal{B}(L^2(H/Z \times H/Z))$, defined by $X\xi(r; r') = \xi(r + r'; r')$, for $\xi \in L^2(G_1 \times G_1)$, and $Y\xi(x, y; x', y') = \xi(x, y; x' - x, y' - y)$, for $\xi \in L^2(H/Z \times G_1)$.

The operator encoding the group actions α and γ is $Z \in \mathcal{B}(L^2(G_2 \times G_1))$, defined by $Z\xi(p, q; r) = \xi \left(p, q - r \sum_{i,j} J_{ij} p_i \mathbf{q_j}; r \right)$. By using partial Fourier transform and the Fourier inversion theorem, we see that it is equivalent to the following (same-named) operator $Z \in \mathcal{B}(L^2(H/Z \times G_1))$:

$$Z\xi(x, y; r) = \xi \left(x + r \sum_{i,j} J_{ij} y_j \mathbf{x}_i, y; r \right), \qquad \text{for } \xi \in L^2(H/Z \times G_1).$$

All this is again very similar to Propositions 3.4 and 3.5, as well as Proposition 3.10 (2).

Using the same strategy as in Proposition 3.9 or in Proposition 3.10, we define our multiplicative unitary operator V_Θ, as follows. In particular, the definition of the cocycle Θ comes directly from the expression of the Poisson bracket given in Proposition 2.3 (3). In the below, $\mathcal{H}$ denotes the Hilbert space $L^2(H/Z \times G_1)$.

Proposition 3.12. [Quantization of Case (3)]. *Define the unitary operator $V \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) = \mathcal{B}(L^2(H/Z \times G_1 \times H/Z \times G_1))$, by $V = (Z_{12} X_{24} Z_{12}^*) Y_{13}$. It is multiplicative. And let $\Theta(x, y, r; x', y', r') := \bar{e}[r'\beta(x, y')] \bar{e} \left[\frac{r'^2}{2} \sum_{i,j} J_{ij} y_j y_i' \right]$, considered as a unitary operator contained in $\mathcal{B}(\mathcal{H} \otimes \mathcal{H})$.*

Then the function Θ is a cocycle for V. In this way, we obtain a multiplicative unitary operator $V_\Theta := V\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$. Specifically

$$V_\Theta \xi(x, y, r; x', y', r')$$

$$= e\left[\frac{r'^2}{2} \sum_{i,j} J_{ij} y_j (y'_i - y_i)\right] \bar{e}[r'\beta(x, y' - y)]$$

$$\xi\left(x - r' \sum_{i,j} J_{ij} y_j \mathbf{x}_i, y, r + r'; x' - x + r' \sum_{i,j} J_{ij} y_j \mathbf{x}_i, y' - y, r'\right).$$

The C^*-bialgebras associated with V_Θ are:

$$S \cong C_0(G_1) \rtimes_\gamma^\sigma (H/Z), \qquad \hat{S} \cong C_0(H/Z) \rtimes_\alpha G_1,$$

together with the comultiplications $\Delta(a) := V_\Theta(a \otimes 1)V_\Theta^*$ for $a \in S$, and $\hat{\Delta}(b) := V_\Theta^*(1 \otimes b)V_\Theta$ for $b \in \hat{S}$. Here, $\sigma : r \mapsto \sigma^r$ is a continuous field of cocycles such that $\sigma^r((x, y), (x', y')) = \bar{e}\left[\frac{r^2}{2} \sum_{i,j} J_{ij} y_j y'_i\right] \bar{e}[r\beta(x, y')]$.

Proof. The multiplicativity of V is a consequence of the fact that (G_1, G_2) forms a matched pair, or equivalently, that G is a group. The function Θ is a cocycle for V, since V_Θ is also multiplicative. The verification of the pentagon equation, $W_{12} W_{13} W_{23} = W_{23} W_{12}$ for $W = V_\Theta$, is straightforward.

As usual, the C^*-bialgebras associated with V_Θ are obtained by

$$S = \{(\omega \otimes \mathrm{id}_\mathcal{H})(V_\Theta) : \omega \in \mathcal{B}(\mathcal{H})_*\}(\subseteq \mathcal{B}(\mathcal{H})),$$

$$\hat{S} = \{(\mathrm{id}_\mathcal{H} \otimes \omega)(V_\Theta) : \omega \in \mathcal{B}(\mathcal{H})_*\}(\subseteq \mathcal{B}(\mathcal{H})).$$

To see the specific C^*-algebra realization of S, consider its typical element $(\omega \otimes \mathrm{id}_\mathcal{H})(V_\Theta)$, where $\omega \in \mathcal{B}(\mathcal{H})_*$. Without loss of generality, we may assume that $\omega = \omega_{\xi,\eta}$, for $\xi, \eta \in \mathcal{H}$. (We may even assume that ξ and η are continuous functions having compact support.) It is a rather standard notation, and is defined by $\omega_{\xi,\eta}(T) = \langle T\xi, \eta \rangle$, for $T \in \mathcal{B}(\mathcal{H})$. It is known that linear combinations of the $\omega_{\xi,\eta}$ are (norm) dense in $\mathcal{B}(\mathcal{H})_*$. Now for $\zeta \in \mathcal{H}$, we have:

$$((\omega_{\xi,\eta} \otimes \mathrm{id}_\mathcal{H})(V_\Theta))\zeta(x, y, r)$$

$$= \int (V_\Theta(\xi \otimes \zeta))(\tilde{x}, \tilde{y}, \tilde{r}; x, y, r)\overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})} \, d\tilde{x} \, d\tilde{y} \, d\tilde{r}$$

$$= \int e\left[\frac{r^2}{2} \sum_{i,j} J_{ij} \tilde{y}_j (y_i - \tilde{y}_i)\right] \bar{e}[r\beta(\tilde{x}, y - \tilde{y})]\xi\left(\tilde{x} - r \sum_{i,j} J_{ij} \tilde{y}_j \mathbf{x}_i, \tilde{y}, \tilde{r} + r\right)$$

$$\overline{\eta(\tilde{x}, \tilde{y}, \tilde{r})}\zeta\left(x - \tilde{x} + r \sum_{i,j} J_{ij} \tilde{y}_j \mathbf{x}_i, y - \tilde{y}, r\right) d\tilde{x} \, d\tilde{y} \, d\tilde{r}$$

$$= \int \bar{e} \left[\frac{r^2}{2} \sum_{i,j} J_{ij} \tilde{y}_j (y_i - \tilde{y}_i) \right] \bar{e}[r\beta(\tilde{x}, y - \tilde{y})] \xi(\tilde{x}, \tilde{y}, \tilde{r} + r)$$

$$\overline{\eta \left(\tilde{x} + r \sum_{i,j} J_{ij} \tilde{y}_j \mathbf{x}_i, \tilde{y}, \tilde{r} \right)} \zeta(x - \tilde{x}, y - \tilde{y}, r) \, d\tilde{x} \, d\tilde{y} \, d\tilde{r}$$

$$= \int F(\tilde{x}, \tilde{y}, r) \sigma^r((\tilde{x}, \tilde{y}), (x - \tilde{x}, y - \tilde{y})) \zeta(x - \tilde{x}, y - \tilde{y}, r) \, d\tilde{x} \, d\tilde{y},$$

where $F(x, y, r) = \int \xi(x, y, \tilde{r} + r) \overline{\eta \left(x + r \sum_{i,j} J_{ij} y_j \mathbf{x}_i, y, \tilde{r} \right)} \, d\tilde{r}$, which is a continuous function since ξ and η are L^2-functions. And

$$\sigma^r((x, y), (x', y')) = \bar{e} \left[\frac{r^2}{2} \sum_{i,j} J_{ij} y_j y_i' \right] \bar{e}[r\beta(x, y')].$$

It immediately follows from these observations that:

$$S \cong \overline{\{(\omega \otimes \mathrm{id}_{\mathcal{H}})(V_\Theta) : \omega \in \mathcal{B}(\mathcal{H})_*\}}^{\| \ \|} \cong C_0(G_1) \rtimes_\gamma^\sigma (H/Z),$$

which is the twisted crossed product algebra with (trivial) action γ, and whose twisting is given by the cocycle $\sigma : r \mapsto \sigma^r$.

Similar computation as above (and similar also to the case of $\hat{A}$ in Proposition 3.9 and of $\hat{A}_V$ in Proposition 3.7) shows that $\hat{S} \cong C_0(H/Z) \rtimes_\alpha G_1$, which is the crossed product algebra with action α, given by $\alpha_r(x, y) = \left(x + r \sum_{i,j} J_{ij} y_j \mathbf{x}_i, y \right)$. $\square$

Essentially, (S, Δ) is a "quantized $C^*(H)$" or a "quantized $C_0(G)$". For instance, if $J \equiv 0$, then we have: $S \cong C^*(H)$. Let us also look at the comultiplication Δ of S below, which shows that it reflects the group multiplication law on G.

Proposition 3.13. *For $\phi \in C_c(G)$, define $L_\phi \in \mathcal{B}(\mathcal{H})$ be defined by*

$$L_\phi \zeta(x, y, r) := \int \phi^\vee(\tilde{x}, \tilde{y}, r) \sigma^r((\tilde{x}, \tilde{y}), (x - \tilde{x}, y - \tilde{y})) \zeta(x - \tilde{x}, y - \tilde{y}, r) \, d\tilde{x} \, d\tilde{y},$$

where σ is the cocycle as in Proposition 3.12, and $\phi^\vee$ denotes the (partial) Fourier transform of ϕ. Namely, $\phi^\vee(x, y, r) = \int \phi(p, q, r) e[p \cdot x + q \cdot y] \, dp \, dq$. We know from the proof of Proposition 3.12 that $S \cong \overline{L(C_c(G))}^{\| \ \|}$, as a C^*-algebra.

The comultiplication, Δ, on S is given by $\Delta(a) = V_\Theta(a \otimes 1)V_\Theta^*$ for $a \in S$. For $\phi \in C_c(G)$, this becomes: $\Delta(L_\phi) = (L \otimes L)_{\Delta(\phi)}$, where $\Delta(\phi) \in C_b(G \times G)$ is the function de-

fined by

$$(\Delta(\phi))(p, q, r; p', q', r') = \phi\left(p + p', q + q' + r'\sum_{i,j} J_{ij} p_i \mathbf{q}_j, r + r'\right).$$

Proof. Write $L_\phi = \int (\mathcal{F}^{-1}\phi)(\tilde{x}, \tilde{y}, \tilde{z}) L_{\tilde{x}, \tilde{y}, \tilde{z}} \, \mathrm{d}\tilde{x} \, \mathrm{d}\tilde{y} \, \mathrm{d}\tilde{z}$, where $\mathcal{F}^{-1}\phi \in C_c(H)$ is the (inverse) Fourier transform of ϕ. Then $L_{\tilde{x}, \tilde{y}, \tilde{z}} \in \mathcal{B}(\mathcal{H})$ is such that

$$L_{\tilde{x}, \tilde{y}, \tilde{z}} \xi(x, y, r) = \bar{e}(r\tilde{z}) \sigma^r((\tilde{x}, \tilde{y}), (x - \tilde{x}, y - \tilde{y})) \xi(x - \tilde{x}, y - \tilde{y}, r).$$

Comparing with the definition of L_ϕ given above, we may regard $L_{\tilde{x}, \tilde{y}, \tilde{z}}$ as $L_{\tilde{x}, \tilde{y}, \tilde{z}} = L_F$, where the function $F \in C_b(G)$ is such that: $F(p, q, r) = \bar{e}[p \cdot \tilde{x} + q \cdot \tilde{y} + r\tilde{z}]$. Actually, $L_{\tilde{x}, \tilde{y}, \tilde{z}}$ is contained in the multiplier algebra $M(S)$. In a sense, the operators $L_{\tilde{x}, \tilde{y}, \tilde{z}}$ for $(\tilde{x}, \tilde{y}, \tilde{z}) \in H$, form the building blocks for the "regular representation" L (or equivalently, for C^*-algebra S).

For $\zeta \in \mathcal{H}$, we have:

$$(\Delta(L_{\tilde{x}, \tilde{y}, \tilde{z}}))\zeta(x, y, r; x', y', r')$$

$$= V_\Theta(L_{\tilde{x}, \tilde{y}, \tilde{z}} \otimes 1)V_\Theta^* \zeta(x, y, r; x', y', r')$$

$$= \bar{e}[(r + r')\tilde{z}]\bar{e}\left[\frac{r^2}{2}\sum_{i,j} J_{ij}\tilde{y}_j(y_i - \tilde{y}_i)\right]\bar{e}[r\beta(\tilde{x}, y - \tilde{y})]$$

$$\bar{e}\left[\frac{r'^2}{2}\sum_{i,j} J_{ij}\tilde{y}_j(y_i' - \tilde{y}_i)\right]\bar{e}[r'\beta(\tilde{x}, y' - \tilde{y})]\bar{e}\left[rr'\sum_{i,j} J_{ij}\tilde{y}_j(y_i - \tilde{y}_i)\right]$$

$$\zeta\left(x - \tilde{x} - r'\sum_{i,j} J_{ij}\tilde{y}_j\mathbf{x}_i, y - \tilde{y}, r; x' - \tilde{x}, y' - \tilde{y}, r'\right).$$

Meanwhile, consider $\Delta(F) \in C_b(G \times G)$, given by

$$(\Delta(F))(p, q, r; p', q', r')$$

$$= \bar{e}\left[(p + p') \cdot \tilde{x} + (q + q') \cdot \tilde{y} + r'\sum_{i,j} J_{ij} p_i \tilde{y}_j + (r + r')\tilde{z}\right].$$

Then by a straightforward computation using Fourier inversion theorem, we can see that for $\zeta \in \mathcal{H}$:

$$(L \otimes L)_{\Delta(F)}\zeta(x, y, r; x', y', r') = (\Delta(L_{\tilde{x}, \tilde{y}, \tilde{z}}))\zeta(x, y, r; x', y', r').$$

In other words, $(L \otimes L)_{\Delta(F)} = \Delta(L_F)$. Remembering the definitions, it follows easily that $\Delta(L_\phi) = (L \otimes L)_{\Delta(\phi)}$ for any $\phi \in C_c(G)$, where $\Delta(\phi)$ is as defined above. $\qquad\square$

Remark 3.14. This proposition shows that for $\phi \in C_c(G)$, the comultiplication sends it to $\Delta(\phi) \in C_b(G \times G)$, such that

$$(\Delta(\phi))(p, q, r; p', q', r') = \phi((p, q, r)(p', q', r')),$$

preserving the group multiplication law on G as given in Proposition 2.3 (3). This result supports our assertion made earlier that (S, Δ) is a "quantized $C_0(G)$".

At this moment, the C^*-bialgebra (S, Δ) is just a quantum semi-group. For it to be properly considered as a locally compact quantum group, we need further discussions on maps like antipode or Haar weight (see [14] for general theory). For this, we may follow the methods we used earlier in [12] or [11], taking advantage of the fact that (S, Δ) is a "quantized $C_0(G)$". Meanwhile, by introducing a deformation parameter, we could also show that (S, Δ) is indeed a deformation quantization of the Poisson–Lie group G, in the direction of its Poisson bracket given in Proposition 2.3 (3). [For Case (2), the deformation quantization is carried out in [9].]

In the current paper, though, we will be content to have demonstrated our program, and shown a constructive method of obtaining an appropriate multiplicative unitary operator for the new example (S, Δ).

Meanwhile, notice the similarity between our example (S, Δ) above and the one constructed by Enock and Vainerman in Section 6 of [6]. The methods of construction are rather different between the two. However, looking at the comultiplications and the cocycles involved, we see a strong resemblance. What this means is that the ingredients at the classical level (information about the groups H and G) are more or less the same.

On the other hand, there is a very significant difference. Namely, the example of [6] has the underlying von Neumann algebra isomorphic to the group von Neumann algebra $\mathcal{L}(H) = C^*(H)''$ of H. While in our case, S is isomorphic to a "twisted" crossed product algebra: unless $J \equiv 0$, the C^*-algebra S is not isomorphic to $C^*(H)$.

In the author's opinion, the example (S, Δ) given here has more merit, considering that its Poisson–Lie group counterpart and its multiplicative unitary operator have all been obtained; the relationship between the Poisson bracket and the cocycle bicrossed product construction of the multiplicative unitary operator have been manifested; as well as that the underlying C^*-algebra is built on the framework of twisted crossed product algebras (more general than ordinary group C^*-algebras or group von Neumann algebras).

4. Other examples

In this section, we give more constructions of several other examples of quantum (semi-)groups. Just as in the previous section, each of these examples will have a twisted crossed product as its underlying C^*-algebra, and the construction can be carried out within the framework of cocycle bicrossed products. These examples are actually slight generalizations of the basic examples given in Section 3, and they are considered as coming from Heisenberg-type Lie bialgebras.

Since the examples in this section will not be fundamentally different from the basic examples covered in Section 3, we will try to make discussions rather brief. We do not even plan to say much about the Poisson–Lie group counterparts to the examples. As before, we will give, for the purpose of efficiency, just the appropriate multiplicative unitary operators. This can be done by giving some small modifications to the twisting cocycles we had above. As in Section 3, discussions about correctly establishing the associated C^*-bialgebras as (C^*-algebraic) locally compact quantum groups will be skipped.

4.1. "Mixed" case of Cases (1) and (2)

Consider the Lie group G defined by the multiplication:

$$(p, q, r)(p', q', r') = (e^{\lambda r'} p + p', \ e^{\nu r'} q + q', r + r'),$$

where $\lambda, \nu \in \mathbb{R}$. Note that if $\nu = -\lambda$ or $\nu = \lambda$, it coincides with the group G given in Case (1) or Case (2) of Proposition 2.3, respectively. In fact, the Lie group G above is obtained as a dual Poisson–Lie group of (H, δ_4), where $\delta_4 : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ is the cobracket defined by $\delta_4 = \left(\frac{\lambda - \nu}{2\lambda}\right) \delta_1 + \left(\frac{\lambda + \nu}{2\lambda}\right) \delta_2$ [recall Definition 2.1]. In this sense, it is a "mixed" case of Cases (1) and (2) earlier.

To find the quantum counterpart of G, or equivalently, the multiplicative unitary operator for the quantum (semi-)group, it really boils down to "changing of the cocycles". So as before, let $\mathcal{H}$ be the Hilbert space consisting of L^2-functions in the (x, y, r) variables. Also let $\eta_{(\lambda, \nu)} := \frac{e^{(\lambda + \nu)r} - 1}{\lambda + \nu}$. Then define $V_\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, given by

$$V_\Theta \xi(x, y, r; x', y', r') = (e^{-(\lambda + \nu)r'/2})^n \bar{e}[\eta_{(\lambda, \nu)}(r')\beta(e^{-\lambda r'} x, y' - e^{-\nu r'} y)]$$

$$\xi(e^{-\lambda r'} x, \ e^{-\nu r'} y, r + r'; x' - e^{-\lambda r'} x, y' - e^{-\nu r'} y, r').$$

It is obtained following pretty much the same procedure as in the cases considered in the previous section. As we see below, it determines a twisted crossed product algebra whose twisting cocycle is given by $\sigma^r((x, y), (x', y')) := \bar{e}[\eta_{(\lambda, \nu)}(r)\beta(x, y')]$.

Proposition 4.1. *Let V_Θ be as in the previous paragraph. It is a multiplicative unitary operator. The C^*-bialgebras associated with V_Θ are:*

$$A \cong C_0(G_1) \rtimes^\sigma_\gamma (H/Z), \qquad \hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1,$$

together with the comultiplications $\Delta(a) := V_\Theta(a \otimes 1)V_\Theta{}^*$ for $a \in A$, and $\hat{\Delta}(b) := V_\Theta{}^*(1 \otimes b)V_\Theta$ for $b \in \hat{A}$. Here, G_1 is the abelian group $G_1 = \{r : r \in \mathbb{R}\}$; the action γ is trivial; and the action α is defined by $\alpha_r(x, y) = (e^{\lambda r} x, e^{\nu r} y)$. Finally, $\sigma : r \mapsto \sigma^r$ is a continuous field of cocycles such that

$$\sigma^r : H/Z \times H/Z \ni ((x, y), (x', y')) \mapsto \bar{e}[\eta_{(\lambda, \nu)}(r)\beta(x, y')] \in \mathbb{T}.$$

Proof. Checking the multiplicativity of V_Θ is straightforward. To see the realizations of the C^*-algebras A and $\hat{A}$, we use the same method as in the proof of Proposition 3.12, investigating the operators $(\omega \otimes \mathrm{id})(V_\Theta)$, $\omega \in \mathcal{B}(\mathcal{H})_*$ and $(\mathrm{id} \otimes \omega')(V_\Theta)$, $\omega' \in \mathcal{B}(\mathcal{H})_*$.

In this way, we can show that $A \cong \overline{L(C_c(H/Z \times G_1))}^{\|\ \|} (\subseteq \mathcal{B}(\mathcal{H}))$, where L is the regular representation defined by

$$L_f \xi(x, y, r) = \int f(\tilde{x}, \tilde{y}, r)\bar{e}[\eta_{(\lambda, v)}(r)\beta(\tilde{x}, y - \tilde{y})]\xi(x - \tilde{x}, y - \tilde{y}, r)\,\mathrm{d}\tilde{x}\,\mathrm{d}\tilde{y}.$$

Here $f \in C_c(H/Z \times G_1)$ and $\xi \in \mathcal{H}$. We can see from this observation that the C^*-algebra A is a twisted crossed product algebra, with trivial action and the twisting cocycle given by $\sigma : r \mapsto \sigma^r$.

Similarly, we can also show that $\hat{A} \cong \overline{\rho(C_c(H/Z \times G_1))}^{\|\ \|} (\subseteq \mathcal{B}(\mathcal{H}))$, where ρ is also the regular representation defined by

$$\rho_f \xi(x, y, r) = \int f(x, y, \tilde{r})\xi(\mathrm{e}^{\lambda\tilde{r}}x, \ \mathrm{e}^{v\tilde{r}}y, r - \tilde{r})\,\mathrm{d}\tilde{r},$$

for $f \in C_c(H/Z \times G_1)$ and $\xi \in \mathcal{H}$. From this, it follows easily that $\hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1$, which is the crossed product algebra with action α. $\quad\square$

Similarly as before, (A, Δ) is considered as a "quantized $C^*(H)$" or a "quantized $C_0(G)$". Further discussion about this case will parallel that of Case (2).

4.2. Example of Rieffel's [20]

Let us now allow our group H to have a higher dimensional center, $Z = \{(0, 0, z) : z \in \mathbb{R}^m\}$. Then H will be now $(2n + m)$-dimensional. For convenience, let us keep the same notation and express the group law on H as

$$(x, y, z)(x', y', z') = (x + x', y + y', z + z' + \beta(x, y')).$$

The differences from the definition of H given in Section 2 are that z and z' are now regarded as vectors (for instance, $z = z_1\mathbf{z}_1 + \cdots + z_m\mathbf{z}_m$), and that $\beta(\cdot, \cdot)$ is no longer the inner product. It will be understood as a Z-valued bilinear map. The new group H is still a two-step nilpotent Lie group which closely resembles the Heisenberg Lie group. This is actually the group considered by Rieffel in [20].

Let G be defined by the multiplication law:

$$(p, q, r)(p', q', r') = (\pi(r')p + p', \rho(r')q + q', r + r'),$$

where π and ρ are representations of the group $G_1 = \{(0, 0, r) : r \in \mathbb{R}^m\}$ on the spaces $\{(p, 0, 0) : p \in \mathbb{R}^n\}$ and $\{(0, q, 0) : q \in \mathbb{R}^n\}$, respectively. Let us impose the following "compatibility condition" between π, ρ, and β, as given by Rieffel.

Compatibility condition [20]: assume that $\beta(\pi(r)^t x, \rho(r)^t y) = \beta(x, y)$ and that $(\det(\pi(r)))(\det(\rho(r))) = 1$, for all r, x, y.

This compatibility condition makes the group G closely analogous to our Case (1) earlier, in the sense that G (together with the linear Poisson bracket dual to the Lie bracket on H) becomes the dual Poisson–Lie group of H. If Z is 1-dimensional, the situation will be exactly same as in Case (1). Because of this, the quantization can be carried out in essentially the same way as in Case (1).

We thus obtain the following unitary operator $V_\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, where $\mathcal{H}$ is the Hilbert space consisting of L^2-functions in the (x, y, r) variables:

$$V_\Theta \xi(x, y, r; x', y', r') = \bar{e}[r' \cdot \beta(\pi(-r')^t x, y' - \rho(-r')^t y)]$$
$$\xi(\pi(-r')^t x, \rho(-r')^t y, r + r'; x' - \pi(-r')^t x, y' - \rho(-r')^t y, r').$$

As before, V_Θ is easily proved to be multiplicative, and it again determines two C^*-bialgebras (A, Δ) and $(\hat{A}, \hat{\Delta})$ [Result is analogous to Proposition 3.10]. As C^*-algebras, we will have: $A \cong C^*(H)$, and $\hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1$, where $\alpha_r(x, y) = (\pi(r)^t x, \rho(r)^t y)$. The C^*-algebra A being isomorphic to the group C^*-algebra $C^*(H)$ again reflects the point that the Poisson bracket on G is linear.

The method was different, but (A, Δ) obtained in this way, together with V_Θ, is exactly the example constructed by Rieffel in [20]. It was really among the first examples of quantum groups given by deformation quantization process, and therefore, was the guiding example of all the examples considered in this work and many others.

4.3. A two-step solvable Lie group: non-unimodular case

Let H and G be $(2n + m)$-dimensional groups, defined by the same multiplication laws as in Section 4.2. But this time, we will no longer require the "compatibility condition". To distinguish the current case from the previous example, let us assume that $\beta(\pi(r)^t x, \rho(r)^t y) \neq \beta(x, y)$ and that $(\det(\pi(r)))(\det(\rho(r))) \neq 1$. So the group G is a non-unimodular, (two-step) solvable Lie group.

Then the setting becomes similar to the example given in Section 4.1. Therefore, what we need now is to find the cocycle expression corresponding to $\sigma^r : ((x, y), (x', y')) \mapsto \bar{e}[\eta_{(\lambda, \nu)}(r)\beta(x, y')]$ of Proposition 4.1.

Note however that since Z and G_1 are higher than 1-dimensional, the counterparts to λ and ν are no longer scalars. So it is somewhat awkward to make sense of the expression $\eta_{(\lambda, \nu)}(r) = \frac{e^{(\lambda+\nu)r} - 1}{\lambda + \nu}$. On the other hand, we can get around this problem if we only consider the numerator part of $\eta_{(\lambda, \nu)}(r)$. This means that we are changing the Poisson bracket on G by the factor of $(\lambda + \nu)$. Since λ and ν are fixed, this modification will not affect the Poisson duality between H and G (although we do not give any explicit description of the Poisson bracket here).

Remark 4.2. One drawback is that in so doing, λ and ν lose some of their flavors as deformation parameters for the cobracket. Furthermore, if $\lambda = \nu = 0$, we will no longer have the linear Poisson bracket as before. (We instead obtain a trivial Poisson bracket.)

Nevertheless, since λ and ν are fixed, non-zero constants (not considered as parameters), we do not have to worry about these problems here.

So let us look for a counterpart to the cocycle $\bar{e}[(e^{(\lambda+\nu)r} - 1)\beta(x, y')]$. For this, we will take $\bar{e}[\Sigma(\beta(\pi(r)^t x, \rho(r)^t y') - \beta(x, y'))]$, where $\Sigma() : Z \to \mathbb{R}$ is defined by

$$\Sigma(z_1 \mathbf{z}_1 + z_2 \mathbf{z}_2 + \cdots + z_m \mathbf{z}_m) = z_1 + z_2 + \cdots + z_m.$$

Note that when Z is 1-dimensional, both cocycles clearly agree. Using this, let us write down the following unitary operator $V_\Theta \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ (notice the similarity with the one obtained in Section 4.1):

$$V_\Theta \xi(x, y, r; x', y', r') = |(\det(\pi(-r')))(\det(\rho(-r')))|^{1/2}$$

$$\bar{e}[\Sigma(\beta(x, \rho(r')^t y' - y) - \beta(\pi(-r')^t x, y' - \rho(-r')^t y))]$$

$$\xi(\pi(-r')^t x, \rho(-r')^t y, r + r'; x' - \pi(-r')^t x, y' - \rho(-r')^t y, r').$$

It is again multiplicative, and it thus determines a pair of C^*-bialgebras (A, Δ) and $(\hat{A}, \hat{\Delta})$. As C^*-algebras, we have: $A \cong C_0(G_1) \rtimes_\gamma^\sigma (H/Z)$, and $\hat{A} \cong C_0(H/Z) \rtimes_\alpha G_1$, where γ is the trivial action, $\alpha_r(x, y) = (\pi(r)^t x, \rho(r)^t y)$, and $\sigma : r \mapsto \sigma^r$ is the continuous field of cocycles given by $\sigma^r((x, y), (x', y')) = \bar{e}[\Sigma(\beta(\pi(r)^t x, \rho(r)^t y') - \beta(x, y'))]$. All these computations are done following the same method we have been using so far.

Remark 4.3. Note that the map Σ defined above is none other than the inner product: $\Sigma(z) = z \cdot \mathbf{1}$, where $\mathbf{1} = 1\mathbf{z}_1 + \cdots + 1\mathbf{z}_m$. Of course, there is no particular reason for choosing the vector $\mathbf{1}$ here, and any fixed vector in Z will be sufficient for our purposes. Still, they will all give rise to essentially the same quantum group since we can vary the bilinear map $\beta(\cdot, \cdot)$ to accommodate the changes. A more significant observation is that the map $((x, y), (x', y')) \mapsto \beta(\pi(r)^t x, \rho(r)^t y') - \beta(x, y')$ is already an additive cocycle having values in Z.

The examples (A, Δ) and $(\hat{A}, \hat{\Delta})$ given in this subsection are not necessarily very complicated ones. However, as far as the author knows, these examples have not been studied before. On the other hand, they are really natural generalizations of the examples explored by the author in his previous papers. Meanwhile, even though we are not explicitly investigating Haar weight and the rest of the quantum group structure maps here, we note that unlike the example in Section 4.2 (or [20]), the quantum group (A, Δ) given here will be non-unimodular.

Appendix A. The classical *r*-matrices and the Poisson structures on *H*

In many cases, compatible Poisson brackets on a Lie group (or equivalently, Lie bialgebra structures) are known to arise from certain solutions of the classical Yang–Baxter equation (CYBE), called the *classical r-matrices*. In this Appendix A, we will first give a very brief background discussion on classical *r*-matrices (see [4,3] for more). We will then show that

our three basic Lie bialgebra structures in Definition 2.1 are indeed obtained from some specific classical r-matrices.

In general, let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be an arbitrary element. Define a map $\delta_r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, by

$$\delta_r(X) = \mathrm{ad}_X(r), \qquad X \in \mathfrak{g}. \tag{A.1}$$

Then δ_r is a one-cocycle on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. (Actually, it is a coboundary on $\mathfrak{g}$). The following result holds.

Proposition A.1 (See [4]). *Let $\mathfrak{g}$ be a Lie algebra and let $r \in \mathfrak{g} \otimes \mathfrak{g}$. The map δ_r given by* Eq. (A.1) *defines a Lie bialgebra structure on $\mathfrak{g}$, if and only if the following two conditions are satisfied*:

- $r^{12} + r^{21}$ *is a $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g}$.*
- $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$ *is a $\mathfrak{g}$-invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.*

In this case, $\mathfrak{g}$ is said to be a coboundary Lie bialgebra.

The simplest way to satisfy the second condition of the proposition is to assume that:

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

This is called the *classical Yang–Baxter equation* (CYBE). A solution of the CYBE is called a "classical r-matrix". A coboundary Lie bialgebra structure coming from a solution of the CYBE is said to be *quasitriangular*. If the classical r-matrix further satisfies $r^{12} + r^{21} = 0$ (i.e. it is a skew solution of the CYBE), it is said to be *triangular*. This terminology is closely related with the "quantum" situation and the so-called *(quasitriangular/triangular) quantum universal R-matrices*.

Remark A.2. The quantization problem of triangular and quasitriangular Lie bialgebras is an important topic in the quantum group theory, mostly (but not exclusively) at the quantized universal enveloping algebra (QUE algebra) setting. Practically, these are the Lie bialgebras that are more or less expected to be quantized. Moreover, the "quantum R-matrices" often play interesting roles in the representation theory of the quantum group counterparts to the Poisson–Lie groups (Lie bialgebras). See [4,3].

Let us now turn our attention to the $(2n + 1)$-dimensional Heisenberg Lie group H and the Heisenberg Lie algebra $\mathfrak{h}$, as given in Section 2. Consider also the following "extended" Heisenberg Lie algebra.

Definition A.3. Let $\tilde{\mathfrak{h}}$ be the $(2n + 2)$-dimensional Lie algebra spanned by the basis elements $\mathbf{x}_i, \mathbf{y}_i$ $(i = 1, \ldots, n)$, $\mathbf{z}, \mathbf{d}$, with the brackets

$$[\mathbf{x}_i, \mathbf{y}_j] = \delta_{ij}\mathbf{z}, \qquad [\mathbf{d}, \mathbf{x}_i] = \mathbf{x}_i, \qquad [\mathbf{d}, \mathbf{y}_i] = -\mathbf{y}_i, \qquad \mathbf{z} \text{ is central.}$$

The Lie group corresponding to $\tilde{\mathfrak{h}}$ is the "extended" Heisenberg Lie group $\tilde{H}$. For group multiplication law on $\tilde{H}$, see Example 3.6 of [9] or Section 2.1 of [10].

For this extended Heisenberg Lie algebra, we can find the following solutions of the classical Yang–Baxter equation (CYBE), $r \in \tilde{\mathfrak{h}} \otimes \tilde{\mathfrak{h}}$. The proofs are straightforward. It follows that we now have the Lie bialgebra structures on $\tilde{\mathfrak{h}}$.

Proposition A.4.

(1) *Let $r = \lambda(\mathbf{z} \otimes \mathbf{d} - \mathbf{d} \otimes \mathbf{z})$, $\lambda \neq 0$. Since $\mathrm{span}(\mathbf{z}, \mathbf{d})$ is an abelian subalgebra of $\tilde{\mathfrak{h}}$, it is easy to see that r satisfies the CYBE: $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$. Since r is also a skew solution, we thus obtain on $\tilde{\mathfrak{h}}$ a "triangular" Lie bialgebra structure $\tilde{\delta}_1$, given by $\tilde{\delta}_1(X) := \mathrm{ad}_X(r)$, $X \in \tilde{\mathfrak{h}}$. Specifically,*

$$\tilde{\delta}_1(\mathbf{x}_i) = \lambda(\mathbf{x}_i \otimes \mathbf{z} - \mathbf{z} \otimes \mathbf{x}_i) = \lambda \mathbf{x}_i \wedge \mathbf{z},$$

$$\tilde{\delta}_1(\mathbf{y}_i) = -\lambda \mathbf{y}_i \wedge \mathbf{z}, \qquad \tilde{\delta}_1(\mathbf{z}) = 0, \qquad \tilde{\delta}_1(\mathbf{d}) = 0.$$

(2) *Let $r = 2\lambda(\sum_{i=1}^n (\mathbf{x}_i \otimes \mathbf{y}_i) + \frac{1}{2}(\mathbf{z} \otimes \mathbf{d} + \mathbf{d} \otimes \mathbf{z}))$, $\lambda \neq 0$. We can show that r satisfies the CYBE, and also that $r^{12} + r^{21}$ is $\mathfrak{h}$-invariant. So we obtain a "quasitriangular" Lie bialgebra structure on $\tilde{\mathfrak{h}}$ given by the following $\tilde{\delta}_2$:*

$$\tilde{\delta}_2(\mathbf{x}_i) = \mathrm{ad}_{\mathbf{x}_i}(r) = \lambda \mathbf{x}_i \wedge \mathbf{z}, \qquad \tilde{\delta}_2(\mathbf{y}_i) = \mathrm{ad}_{\mathbf{y}_i}(r) = \lambda \mathbf{y}_i \wedge \mathbf{z},$$

$$\tilde{\delta}_2(\mathbf{z}) = \mathrm{ad}_{\mathbf{z}}(r) = 0, \qquad \tilde{\delta}_2(\mathbf{d}) = \mathrm{ad}_{\mathbf{d}}(r) = 0.$$

Note that $\mathfrak{h}(\subseteq \tilde{\mathfrak{h}})$ is a Lie subalgebra of $\tilde{\mathfrak{h}}$, and also that δ_1 and δ_2 given in Definition 2.1 are obtained by restricting $\tilde{\delta}_1$ and $\tilde{\delta}_2$ above. In other words, $(\mathfrak{h}, \delta_i)$ $(i = 1, 2)$ is a *sub-bialgebra* of $(\tilde{\mathfrak{h}}, \tilde{\delta}_i)$ $(i = 1, 2)$, and hence a Lie bialgebra itself. In this way, we recover the Poisson brackets of Cases (1) and (2).

For Case (3), see the following proposition (proof is again straightforward). In this case, we do not need to introduce the extended Heisenberg Lie algebra. As in Section 2, let (J_{ij}) be a skew, $n \times n$ matrix ($n \geq 2$).

Proposition A.5. *Let $r \in \mathfrak{h} \otimes \mathfrak{h}$ be defined by $r = \sum_{i,j=1}^n J_{ij}\mathbf{x}_i \otimes \mathbf{x}_j$. Since $\mathrm{span}(\mathbf{x}_i : i = 1, 2, \ldots, n) \subseteq \mathfrak{h}$ is an abelian subalgebra, r clearly satisfies the CYBE. It is also a skew solution. Therefore, we obtain a "triangular" Lie bialgebra structure δ_3 on $\mathfrak{h}$:*

$$\delta_3(\mathbf{x}_j) = 0, \qquad \delta_3(\mathbf{y}_j) = \sum_{i=1}^n J_{ij}\mathbf{x}_i \wedge \mathbf{z}, \qquad \delta_3(\mathbf{z}) = 0.$$

Having the knowledge that our Poisson brackets come from certain classical r-matrices is quite useful. For instance, in [9] (concerning Case (2)), we could find an operator R which can be considered as a "quantum universal R-matrix" (the quantum counterpart to the classical r-matrix). Using the operator R in [10], we could show an interesting (genuinely quantum) property of "quasitriangularity" in the representation theory of (A, Δ).

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